d'où pour tout $n \in IN^*$ on a : $a_{n+1} = \dfrac{(2n+2)a_n}{3+2n}$

b) On a : $a_0 = 1$ et $\dfrac{2^{2\times0}(0!)^2}{(2\times0+1)!} = \dfrac{1}{1} = 1$ d'où $a_0 = \dfrac{2^{2\times0}(0!)^2}{(2\times0+1)!}$

Soit $n \in IN$, supposons que : $a_n = \dfrac{2^{2n}(n!)^2}{(2n+1)!}$ et montrons que $a_{n+1} = \dfrac{2^{2(n+1)}((n+1)!)^2}{(2(n+1)+1)!}$.

On a : $a_{n+1} = \dfrac{(2n+2)a_n}{3+2n} = \dfrac{(2n+2)}{3+2n}\dfrac{2^{2n}(n!)^2}{(2n+1)!} = \dfrac{(2n+2)^2.2^{2n}(n!)^2}{(2n+3).(2n+2).(2n+1)!}$

$= \dfrac{2^2.(n+1)^2.2^{2n}(n!)^2}{(2n+3)!} = \dfrac{2^{2n+2}.((n+1)(n!))^2}{(2n+3)!} = \dfrac{2^{2(n+1)}((n+1)!)^2}{(2(n+1)+1)!}$

C) $U_n = \sum_{p=0}^{n}\left(\dfrac{p!}{1\times3\times5\times....\times(2p+1)}\right)$

1°/ $\dfrac{p!}{1\times3\times5\times....\times(2p+1)} = \dfrac{p!(2\times4\times6\times.....\times2p)}{1\times2\times3\times4\times5\times....\times2p\times(2p+1)} =$

$\dfrac{p!((2\times1)\times(2\times2)\times(2\times3)\times.....\times(2\times p))}{(2p+1)!} = \dfrac{p!.2^p.p!}{(2p+1)!} = \dfrac{2^p.(p!)^2}{(2p+1)!}$

On a : $U_n = \sum_{p=0}^{n}\dfrac{2^p.(p!)^2}{(2p+1)!} = \sum_{p=0}^{n}\dfrac{1}{2^p}\dfrac{(2^p.p!)^2}{(2p+1)!} = \sum_{p=0}^{n}\dfrac{1}{2^p}\int_0^1(1-t^2)^p dt =$

$\int_0^1\sum_{p=0}^{n}\dfrac{1}{2^p}(1-t^2)^p dt = \int_0^1\sum_{p=0}^{n}\left(\dfrac{1-t^2}{2}\right)^p dt$

$= \int_0^1\dfrac{1-\left(\dfrac{1-t^2}{2}\right)^{n+1}}{1-\dfrac{1-t^2}{2}}dt = 2.\int_0^1\dfrac{1-\left(\dfrac{1-t^2}{2}\right)^{n+1}}{1+t^2}dt.$

2°/a) On pose $V_n = 2I - U_n = 2.\int_0^1\dfrac{dt}{1+t^2} - 2.\int_0^1\dfrac{1-\left(\dfrac{1-t^2}{2}\right)^{n+1}}{1+t^2}dt$